

# A Theory for the Term Structure of Interest Rates

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**Abstract** The Convolution and Master equations governing the time behavior of the term structure of Interest Rates are particularly simple for continuous variables. The infinitesimal generator of the generalized Markov process is usually a distribution. We believe that the discretised forms of the equation and of the generator are better suited to compare with actual distributions. They help to avoid the Gaussian-like tail behavior generally derived from the continuous equations with a finite number of Markov processes. In this paper, the notion of discretised Seed is introduced which naturally leads to an infinite superposition of Markov processes and hence allows a power (rather than exponential) decrease of the empirical probabilities of the variations of the interest rates. The discretised theoretical distributions of probabilities matching the empirical data from the Federal Reserve System (FRS) are deduced from a discretised seed which enjoys remarkable scaling laws. In particular the tails of the distributions are very well reproduced. These results may be used to develop new methods for the computation of the value-at-risk and fixed-income derivative pricing and suggest the appearance of the critical exponents related to models based on self-organized systems.

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# 1 Introduction

The accuracy of the interest rates variations modelling is an important issue especially in the context of the evaluations of the value-at-risk and marked-to-market positions in trading floors.

In two recent articles [1], [2], it was shown using empirical data published by the governors of the Federal Reserve System [3] from 1962 until 2002 that the term structure of interest rates decreases essentially for large variations of the interest rates as a power and that this power is of the order three to four. Moreover the distributions seem to obey simple approximate scaling laws as functions of the initial interest rate, of the lag and of the maturity. These findings invalidate many models which predict distributions having either very short tails, generally exponentially decreasing, or very long tails as do Levy type structures [5], [6], [7], [8].

In this paper, a theoretical model is built to serve as a basis for computing the distribution of the variation of interest rates in terms of a few fundamental “microscopic parameters” whose meaning will be highlighted. At the basis of the theory, the notion of “seed” (a discretised form of the infinitesimal generator [4]) is introduced. It is closely related to the variation of the interest rates for a very short but finite time intervals. We believe that this discretisation is an important issue, closely linked to the fact that empirical interest rates are, often by regulation, expressed as integer multiples of the basis point and obtained by averages performed on a finite set of discrete times.

In order to simplify the presentation and highlight preceding results available in the literature [4], the problem is set in terms of continuous variables. Later, to allow numerical simulations and come closer to the empirical distributions, the variables are discretised.

All along this article, in order to make a connection to a real situation, we have chosen to refer systematically to an application of our ideas to the FRS data [3]. Needless to say, we expect our analysis to be extendable, *mutatis mutandis*, to many other situations.

## 2 The basic equations with continuous variables

In this section, the basic equations, which govern the continuous time propagation of the term structure of interest rates, are briefly reviewed, analysed and commented from an econo-physicist’s point of view.

Suppose that, the interest rate for a certain maturity  $[m]$  has the value  $I_0$  at

time  $t_0$ . We want to study the normalized density of probability (as seen at time  $t$ )

$$p_t^{[m]}(t_f, I_f, t_0, I_0) \quad (1)$$

that, at a later (final) time  $t_f$ , thus after a lag

$$L = t_f - t_0 , \quad (2)$$

the interest rate has a value  $I_f$ . In principle, this density of probability  $p_t$  is an unknown function of the five “continuous variables”  $t, t_f, I_f, t_0, I_0$ . In order to simplify the notation we will restrict ourselves to a given maturity and suppress the corresponding upper index.

The compounded probability (at time  $t$ )  $\overline{P}_{t,t_f,I_a \leq I_f \leq I_b,t_0,I_0}$  that, starting with an initial interest rate  $I_0$  at the initial time  $t_0$ , the final interest rate  $I_f$  at the final time  $t_f$  is in the interval between  $I_a$  and  $I_b$  is given by the integral on this interval of the probability density

$$\overline{P}_{t,t_f,I_a \leq I_f \leq I_b,t_0,I_0} = \int_{I_a}^{I_b} p_t(t_f, I_f, t_0, I_0) dI_f \quad (3)$$

with, obviously, by normalization

$$\overline{P}_{t,t_f,-\infty \leq I_f \leq +\infty,t_0,I_0} = 1 . \quad (4)$$

We will now make the hypothesis that the interest rate variations satisfy some market laws which are rather stable and are governed by sufficiently smooth equations. Let us try to state more precisely and justify our simplifying assumptions.

## 2.1 Time translation invariance

This is a very delicate hypothesis. It is equivalent in saying that, whatever be the political or economical situation the average behavior will be identical. The effects of the exceptional situations (political or economical crisis) which usually may lead to seemingly incoherent variations of the interest rates are accurately taken into account by the tails of the functions which are used. The postulate is that, even if there are extreme situations of various importance, all in all they connect smoothly with the more normal situations which prevail during the peaceful times. It is precisely these extreme situations which are the prime reason for the fat tails of the distributions. And they must be incorporated correctly by the model. To repeat, the hypothesis is that what happens in the exceptional situations is well taken into

account by the smooth tails of the distributions. There is no abrupt transition between really exceptional situations and what we would call the normal situations. Between the extremes, there are situations of intermediate seriousness which lead for the distribution to a smooth passage from a restful period to a chaotic one. But let us make our arguments more precise.

Mathematically, for any of the times  $t, t_0, t_f$ , the global time translation operation is given by

$$t' = t + T \quad (5)$$

$$t'_0 = t_0 + T \quad (6)$$

$$t'_f = t_f + T . \quad (7)$$

where  $T$  is the value of the translation time (minutes or days or months or years later). Suppose that at some time  $t'_0$  later than  $t_0$  given by the translation time  $T$  the interest rate  $I'_0$  is again exactly equal to  $I_0$ . Technically, the time translation invariance demands that the densities of probability to obtain the same final interest rate  $I'_f = I_f$  be equal

$$p_{t'}(t'_f, I_f, t'_0, I_0) = p_t(t_f, I_f, t_0, I_0) \quad (8)$$

whatever be  $T$ . The financial interpretation of this equality is that the market laws are stable in time. If the same situation is reproduced later, it will evolve with the same probabilities.

It is not difficult to prove mathematically that this condition essentially implies that the density of probability does not depend on the three time variables  $t, t_0, t_f$  separately but only on two of their differences, say  $t - t_0$  and the lag

$$L = t_f - t_0 . \quad (9)$$

Let us finally discuss the dependence in  $t$  which is the time at which the information is needed. Obviously if  $t$  is larger than  $t_0$ , in principle all the information of the what the actual interest rates have been between  $t_0$  and  $t$  is available and known. The related probabilities are nor useful any more. If  $t$  is smaller than  $t_0$  it means that we want to evaluate the probabilities for evolutions of the interest rates happening at later times. Again, if the market laws are stable, the evolution during a fixed lag  $L$  as seen at the time( $t = t_0$ ) or at a earlier time ( $t < t_0$ ) should be identical. The same initial state ( $I_0$  at  $t_0$ ) should lead statistically to the same final state ( $I_f$  at  $t_f$ ). This means invariance under a further logically different time transformation

(5) while  $t_0$  and  $t_f$  are kept fixed

$$t' = t + T \quad (10)$$

$$t'_0 = t_0 \quad (11)$$

$$t'_f = t_f \quad (12)$$

This implies that the density of probability is independent of its index  $t$ .

Gathering the result of the two invariances, we find the time invariant restricted form of the density

$$p_t(t_f, I_f, t_0, I_0) = \tilde{p}(t_f - t_0, I_f, I_0) . \quad (13)$$

In other words, it does not depend independently on  $t$ ,  $t_f$  and  $t_0$  but on the lag only.

The invariance under the two continuous time translations thus implies that the density of probability  $p_t$  is reduced to a function of three continuous variables only: the lag, the final interest rates  $I_f$  and initial interest rate  $I_0$ . The variation  $V$  of the interest rate during the lag  $L$  is defined as

$$V = I_f - I_0 . \quad (14)$$

For later convenience, a last variable change is performed to define the time translation invariant density of probability  $p$  as a function of the lag, of the initial interest rate and of the variation of the interest rate. From now on, the form

$$p(L, V, I_0) \equiv \tilde{p}(t_f - t_0, V + I_0, I_0) \quad (15)$$

will be used as the basic interest rate distributions.

## 2.2 The normalization of the probability

The total probability (4) that after the lag  $L$  the rate  $I_f$  has any value must be equal to one. This implies the normalization

$$\int_{-\infty}^{+\infty} p(L, I_f - I_0, I_0) dI_f \equiv \int_{-\infty}^{+\infty} p(L, V, I_0) dV = 1 . \quad (16)$$

This normalization should hold whatever be the lag and whatever be the initial rate.

## 2.3 The composition of the probabilities

The basic equations, which govern the composition of the probability densities  $p$  (essentially the Chapman-Kolmogoroff equations [4]), are well-known. Let us write

them in our notation. Intuitively, consider three times  $t_0, t_i, t_f$  where the intermediate time  $t_i$  lies between the initial time  $t_0$  and the final time  $t_f$ .

$$t_0 \leq t_i \leq t_f . \quad (17)$$

The initial rate is  $I_0$ .

During the time lag  $L_1 = (t_i - t_0)$  the interest rate has a density of probability  $p(t_i - t_0, I_i - I_0, I_0)$  to reach the intermediate value  $I_i$ . Then starting from the intermediate time  $t_i$  up to final time  $t_f$  (i.e. during the second time lag  $L_2 = (t_f - t_i)$ ) the intermediate observed interest rate  $I_i$  at time  $t_i$  has a density of probability  $p(t_f - t_i, I_f - I_i, I_i)$  to become  $I_f$  at the final time  $t_f$ . Since the interest rate at the intermediate time  $t_i$  can take any value, the density of probability starting from the rate  $I_0$  at initial time  $t_0$  to end up with a rate  $I_f$  at the final time  $t_f$  is given by the integration on  $I_i$  at the intermediate time (convolution of the probabilities)

$$p(t_f - t_0, I_f - I_0, I_0) = \int_{-\infty}^{+\infty} p(t_f - t_i, I_f - I_i, I_i) p(t_i - t_0, I_i - I_0, I_0) dI_i . \quad (18)$$

This is the basic equation which the probability distribution has to fulfil. It shows how the probability distributions of two successive lags  $L_1 = (t_i - t_0)$  and  $L_2 = (t_f - t_i)$  compose to form the probability distribution for the lag  $L = t_f - t_0 = L_1 + L_2$ . Eq.(18) should hold whatever be the intermediate time.

It is convenient to rewrite the equation by using the new variables  $I, V$  and the new integration variable  $W$

$$\begin{aligned} I &= I_0 & , & & I_0 &= I \\ V &= I_f - I_0 & , & & I_f &= V + I \\ W &= I_i - I_0 & , & & I_i &= W + I \\ L_1 &= t_i - t_0 & , & & L_2 &= t_f - t_i \end{aligned} \quad (19)$$

as

$$p(L_1 + L_2, V, I) = \int_{-\infty}^{+\infty} p(L_2, V - W, I + W) p(L_1, W, I) dW \quad (20)$$

This is the basic equation.

## 2.4 Initial conditions

The probability distribution has to satisfy an initial condition which can be described in terms of the Dirac  $\delta$  distribution (see Appendix A). Indeed if the initial rate at time  $t_0$  has the value  $I_0$  (whatever  $I_0$  is), at  $t_f = t_0$  (i.e. after a zero lag  $L = t_f - t_0 =$

0) we know for sure that the rate is still  $I_0$ . The density of probability for the final rate to be different from  $I_0$  is zero. In other words, when the lag is zero, the rate has not moved. It is still  $I_0$  with probability one.

Mathematically, this implies that

$$p(0, V, I_0) = \delta(V) . \quad (21)$$

Some properties of the distribution  $\delta(V)$  are given in Appendix (A) together with a few useful approximations in terms of more conventional functions which will be used later. In particular, the Dirac distribution is normalized

$$\int_{-\infty}^{+\infty} p(0, V, I_0) dV \equiv \int_{-\infty}^{+\infty} \delta(V) dV = 1 . \quad (22)$$

As it should, in agreement with the composition of probabilities (18) evaluated either for  $t_i = t_0$  or for  $t_i = t_f$ , the Dirac distribution, composed with any distribution, satisfies the identities

$$\begin{aligned} p(t_f - t_0, I_f - I_0, I_0) &= \int_{-\infty}^{+\infty} p(t_f - t_0, I_f - I_i, I_i) p(0, I_i - I_0, I_0) dI_i \\ &= \int_{-\infty}^{+\infty} p(0, I_f - I_i, I_i) p(t_f - t_0, I_i - I_0, I_0) dI_i \end{aligned} \quad (23)$$

showing again that an evolution during a zero lag is, in fact, not an evolution.

## 2.5 The seed

By using the convolution equation (18), if one knows the probability distributions  $p(\epsilon, V, I_0)$  for a given lag  $L = \epsilon$ , whatever be the value of  $\epsilon$ , the probability distribution for  $p(2\epsilon, V, I_0)$  for a lag of  $2\epsilon$  can easily be computed by the convolution of  $p(\epsilon, V, I_0)$  with itself. Then  $p(3\epsilon, V, I_0)$  is obtained by the convolution of  $p(2\epsilon)$  with  $p(\epsilon, V, I_0)$ . By successive iterations  $p(n\epsilon, V, I_0)$  can be computed for any positive integer  $n$ . Hence, if one knew the distribution of probability for a very small lag,  $\epsilon$  much smaller than the empirical lags, the distribution for a lag  $L$  much larger than  $\epsilon$  could be computed approximately by simple successive integration. Essentially a number of integration equal to the integer  $n$  closest to  $(L/\epsilon - 1)$ . This approximation would become better and better when  $\epsilon \rightarrow 0$ .

It is then tempting to let  $\epsilon$  go to zero and to take the first order approximation, which we call the seed  $S(V, I_0)$  for reasons to follow, as

$$\begin{aligned} S(V, I_0) &= \lim_{\epsilon \rightarrow 0} \frac{p(\epsilon, V, I_0) - p(0, V, I_0)}{\epsilon} \\ &= \partial_L p(L, V, I_0) \Big|_{L=0} . \end{aligned} \quad (24)$$

Since the  $V$ -integration of the two  $p$  terms in the right hand side are equal (before  $\epsilon$  is put to zero) by (16), the seed  $S$  satisfies the normalization condition

$$\int_{-\infty}^{\infty} S(V, I_0) dV = 0 . \quad (25)$$

This seed is essentially related to what is often called the "infinitesimal generator" of a Markov process in analogy with the infinitesimal generator familiar in the theory of Lie groups and semi-groups. It allows, in principal, the computation of the time evolution for a finite time from the knowledge of what has happened during an infinitesimal small amount of time.

We have however decided to call it seed for two reasons. First, all the variables (interest rates and times) are discretised. Thus, the meaning of the seed does not coincide with the meaning of the infinitesimal generator as it does not correspond to an infinitesimal time anymore. Second, Markov processes are usually based on random walk models. For a finite superposition of Markov processes (arbitrary large but finite) the distribution of probability turns out to be a finite superposition of exponentials (short tails) (see (3)). Our generalisation will in fact be equivalent to an infinite superposition and hence will transcend exponentials and hence escape the ill-fated short tails.

If a Taylor expansion is used, in first approximation, for  $L = \epsilon$  very small, the probability distribution  $p$  can be written

$$p(\epsilon, V, I_0) = p(0, V, I_0) + \epsilon S(V, I_0) . \quad (26)$$

In the continuous case, this is a distribution. Nevertheless in the discretised case, this form, with function rather distributions, can be used as a very good approximation.

In the rest of the section, we discuss a few facts related to the presence of distributions rather than functions. The reader which is not interested in these mathematical considerations should jump immediately to section (4). Indeed, technically, it should be stressed that, as the initial condition (21) is a distribution rather than a function, we should expect that the seed is also a distribution with support restricted to  $V = 0$ .

## 2.6 The Master Equation. Continuous variables

If the seed is known as a distribution, the density of probability of the interest rate variation can easily be computed by integrating the master equation as an integro-differential equation

$$\partial_L p(L, I_f - I_0, I_0) = \int_{-\infty}^{+\infty} p(L, I_f - I_i, I_i) S(I_i - I_0, I_0) dI_i \quad (27)$$



which follows from the convolution equation (18). With the change of variables (19) the master equation can equivalently be written

$$\partial_L p(L, V, I) = \int_{-\infty}^{+\infty} p(L, V - W, I + W) S(W, I) dW . \quad (28)$$

Technically, one obtains Eq.(27) by differentiating the convolution equation (18) with respect to  $t_i$  and letting  $t_i \rightarrow t_0$ . The master equation (27) is a consequence of the convolution equation. Conversely, in Appendix (B), it is shown, formally, that the solutions of the master equation satisfy the convolution equation (18).

### 3 The Gauss distribution as the solution for the simplest seed

The simplest possible (67) (see Appendix (A)) seed is the distribution

$$S(V, I_0) = \kappa(I_0) \partial_V^2 \delta(V) . \quad (29)$$

This can be justified naïvely as follows. During a very small amount of time  $\epsilon$  one expects the variation of the interest rate  $V$  to be very small, say at most of the order  $\rho$  where  $\rho$  decreases and goes to zero with  $\epsilon$ . Introduce the three intervals of length  $\rho$ , the left interval  $L = [-3\rho/2, -\rho/2]$ , the center one  $C = [-\rho/2, +\rho/2]$  and the right one  $R = [+ \rho/2, +3\rho/2]$ . Outside these three intervals the probability is essentially zero as is the density of probability. In the left and right intervals the probability is small say of the order  $\kappa$  and in the center it must be  $1 - 2\kappa$  to conserve the normalization of the probability. If the density of probability is then supposed to be constant in the three intervals, the seed becomes  $\kappa/\rho$  in the left and right intervals and  $-2\kappa/\rho$  in the center interval. Letting the time interval  $\epsilon$  go to zero induces  $\rho$  to go to zero and the limiting seed becomes, up to  $\kappa$ , the second derivative of the  $\delta$  distribution. A Markov process is a *finite* superposition of terms of the form (29) with possible higher (even) derivatives of the  $\delta$  distribution.

Taking this simplest form of the seed, the integro-differential equation (27) becomes a much simpler partial differential equation

$$\partial_L p(L, I_f - I_0, I_0) = \kappa(I_0) \partial_{I_0}^2 p(L, I_f - I_0, I_0) \quad (30)$$

which is also written

$$\partial_L p(L, V, I_0) = \kappa(I_0) \left( \partial_V^2 p(L, V, I_0) - 2 \partial_V \partial_{I_0} p(L, V, I_0) + \partial_{I_0}^2 p(L, V, I_0) \right) . \quad (31)$$

This equation is still rather complicated.

It is shown in Appendix (C), assuming no  $I_0$  dependence and simplified scaling laws (83) and (84), that the solution of the master equation for the seed (29) is of Gaussian type

$$\tilde{p}(\tilde{V}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\tilde{V}^2}{2\sigma^2}} . \quad (32)$$

This solution as highlighted in [1] and [2] is completely excluded by the observations as are all Markov's approaches which lead to finite superpositions of such Gaussians.

## 4 Some considerations about the discretisation of the problem. Numerical integration

### 4.1 Why discretisation?

The choice of discretising the distributions is not done here for convenience but is motivated by the discrete nature of the available data and the way they are produced by the laws of supply and demand. Naturally, market makers are always working in units of “something, 1/32 of \$, basis point . . . . Thus, the natural variation measure units are by nature discrete. The same discussion applies to Federal Reserve System data [3]. In this case, we are quite naturally led to discretise both the time variable and the interest rates variable. As a consequence the master equation be better discretised. Indeed, first, by regulation, the interest rates are expressed by integers in basis point ( $10^{-2}$  percent) and hence naturally discrete. Then, usually, the determination of the daily average interest rates is compounded by a finite set of individual contracts finalized at certain times. These times do not occur at fixed intervals. But they are clearly not continuous and probably better taken into account by a set of discrete times at regular intervals rather than by a continuous time. This is a critical point. We show that the discretisation leads to extremely accurate results.

Moreover, technically, given a seed which is a distribution, the convolution equation (20) or the master equation (28) are usually very difficult or impossible to solve analytically. Instead, in this present work, a numerical procedure has been implemented. This numerical integration depends on the discretisation of the continuous equation using discretised variables on a grid.

We have focussed our attention to the following specific form of the continuous

equations (20) for  $L = L_1 = L_2$

$$p(2L, V, I) = \int_{-\infty}^{+\infty} p(L, V - W, I + W) p(L, W, I) dW . \quad (33)$$

In the following, the ideas and the delicate points are briefly described. The precise application of the formalism to the FRS data [3] and the results are summarized and highlighted in section (7).

In this section are presented in turn the ideas which are needed to perform the approximate numerical integration of the equation. The continuous equations are discretised by associating a finite lattice (a grid) to the continuous variables. The initial conditions and the seed are then defined on a finite set of points at the nodes of the grid. This allows then to define the  $\chi$ -squared merit function.

## 4.2 The ideas behind the grid

The equation (33) depends on three continuous variables. In order to perform numerical integrations, the three variables have to be discretised.

The Lag grid

Suppose that the distribution is known for some value  $s_L$  of the lag  $L$ . From the first application of (33) one obtains the distribution for  $L = 2s_L$ . Applying (33) a second time allows the determination of the distribution for  $4s_L \equiv 2^2 s_L$ . The iteration of the procedure produces the distribution for any  $L$  of the form  $2^l s_L$  i.e. on a the  $L$ -grid of values

$$L = 2^l \cdot s_L \quad l = 0, \dots, N_L . \quad (34)$$

Our choice of the Lag grid

We have chosen to obtain a lag  $L = 1$  day after  $N_L = 10$  iteration. This leads to a value of

$$s_L = \frac{1}{2^{10}} \text{ days} \quad (35)$$

i.e. about 30 seconds. This is a perfect choice if individual contracts are concluded at this rate. If it turns out that the lapse between two successive contracts is smaller,  $N_L$  should be increased.

The  $V$  grid

For the  $V$  variable, the best choice is a regularly spaced grid consisting of points of the form

$$V = v \cdot s_V \quad (36)$$

where  $s_V$  is the step size in  $V$  and  $v$  is an integer. A smaller  $s_V$  should lead to a choice of a smaller  $s_V$ .

Our choice of the  $V$  grid

We have chosen to take  $s_V$  to be

$$s_V = \left(\frac{1}{1.4}\right)^{10} \text{ basis point} \quad (37)$$

i.e. around  $(\frac{1}{\sqrt{2}})^{10}$  basis point. Indeed a larger  $s_V$  turned out not to be small enough while a smaller one was not necessary and increased the computation time and memory requirements with little reward.

The  $I_0$  grid

For the  $I_0$  variable, the best choice is a regularly spaced grid consisting of points of the form

$$I_0 = i \cdot s_I \quad (38)$$

where  $s_I$  is the step size in  $I_0$  and  $i$  is an integer.

Our choice of the  $I_0$  grid

Since all the parameters vary with  $I_0$  rather weakly, a choice of

$$s_I = \text{one percent} \quad (39)$$

is quite sufficient.

In our actual computations due memory size problems, we have been led to adapt the  $s_V$  step size to the level  $l$  of the iteration. The step size is progressively increased in such a way as to become one basis point at the tenth iteration. This required delicate numerical adjustments

### 4.3 The ideas behind the discretised equations

Obviously, on the grid (34), (36), (38) the equation (33) becomes

$$p(2^{l+1}s_L, v s_V, i s_I) \approx s_V \left\{ \sum_{w=-\infty}^{\infty} p(2^l s_L, (v-w)s_V, i s_I + w s_V) \times \right. \\ \left. \times p(2^l s_L, w s_V, i s_I) \right\} . \quad (40)$$

To be on the grid, the argument  $i s_I + w s_V$  which appears in the first  $p$  on the right hand side must be an integer multiple of  $s_I$ . This would at first sight imply that  $s_V$  is an integer multiple of  $s_I$ . In fact, our numerical implementation uses, for technical reasons, a more elaborate grid than (36) in the  $V$  and a better adapted form of the convolution (40) together with smoothing and interpolation techniques. Moreover the normalization implies

$$\sum_{v=-\infty}^{\infty} p(2^l s_L, v s_V, i s_I) = 1 . \quad (41)$$

### 4.4 The ideas behind the discretised initial condition

The initial condition (21) on the grid is taken as a step-type (69) approximation of the Dirac  $\delta$  distribution, as explained in Appendix (A). Namely

$$p(0, 0, i s_I) = \frac{1}{s_V} , \\ p(0, v s_V, i s_I) = 0 \quad \text{for } v \neq 0 . \quad (42)$$

### 4.5 The ideas behind the discretised seed

Using the discretisation initial condition, for  $\epsilon = s_L$  chosen sufficiently small, the Taylor expansion (26) of  $p$  to the first order in  $s_L$

$$p(s_L, v s_V, i s_I) = p(0, v s_V, i s_I) + s_L S(v s_V, i s_I) . \quad (43)$$

The resulting  $p(s_L, v s_V, i s_I)$  must be a true function (not a distribution anymore). It must be positive and normalized (41)

$$\sum_{v=-\infty}^{\infty} S(v s_V, i s_I) = 0 . \quad (44)$$

Hence,  $S(v_{s_V}, i_{s_I})$  is expected to be a step-type function of  $v_{s_V}$  with the following properties

$$\begin{aligned} S(0, i_{s_I}) &= - \sum_{v \neq 0} S(v_{s_V}, i_{s_I}) \\ S(v_{s_V}, i_{s_I}) &> 0 \quad \text{for } v \neq 0 \\ S(0, i_{s_I}) &< \frac{1}{s_L s_V} . \end{aligned} \tag{45}$$

The form of the seed inferred by the FRS data is presented and discussed in section (5).

## 4.6 The ideas behind the $\chi$ -squared function

In order to determine the seed free parameters, a  $\chi$ -squared minimization method is used.

As usual the generic  $\chi$ -squared function is defined by

$$\chi^2 = \sum_k \frac{(N_{\text{theory}}(k) - N_{\text{data}}(k))^2}{\sigma(k)^2} \tag{46}$$

where the sum is performed on the generic discrete label  $k$  indexing all the available data. In (46),  $N_{\text{data}}(k)$  and  $N_{\text{theory}}(k)$  are respectively the number of events observed and predicted for the label  $k$ .

The natural error  $(\sigma(k)_{\text{natural}})$  is usually taken as if the distribution was of Poisson type i.e.

$$\begin{aligned} \sigma(k)_{\text{natural}} &= \sqrt{N_{\text{data}}(k)} & \text{if } N_{\text{data}}(k) \neq 0 \\ \sigma(k)_{\text{natural}} &= 1 & \text{if } N_{\text{data}}(k) = 0 . \end{aligned} \tag{47}$$

The choice of the error function requires more attention in the present case. As has been already stressed, the theory should take into account and reproduce as accurately as possible the tails of the experimental distributions. This is achieved by choosing in the  $\chi$ -squared a modified form of  $\sigma(k)$

$$\begin{aligned} \sigma(k)_{\text{modified}} &= (N_{\text{data}}(k) N_{\text{theory}}(k))^{1/4} & \text{if } N_{\text{data}}(k) \neq 0 \\ \sigma(k)_{\text{modified}} &= (N_{\text{theory}}(k))^{1/4} & \text{if } N_{\text{data}}(k) = 0 . \end{aligned} \tag{48}$$

Obviously, if theory and experiment match very closely the  $\sigma$ 's in (47) and (48) are essentially identical. This happens in the bulk of the distribution. For the tails

| $[m]$     | [1]   | [2]  | [3]   | [5]   | [7]  | [10]  | [20] | [30] |
|-----------|-------|------|-------|-------|------|-------|------|------|
| $N_{eve}$ | 10698 | 7106 | 10698 | 10698 | 8828 | 10698 | 9009 | 6243 |

Table 1: The number of events  $N_{eve}$  for the maturities  $[m]$ .

of the distributions, when  $N_{data}$  is equal to zero or one event, the  $\sigma(k)_{modified}$  is smaller than  $\sigma(k)_{natural}$ . Hence, more emphasis is put on these tail points by the minimization procedure. The tests performed using both forms of  $\sigma(k)$  confirm this argument. The use of (48) leads to a very good agreement between theory and data in the tails as can be seen in the Figures.

## 5 The Federal Reserve System data. Determination of $N_{data}(v, i_{bin})$ for each allowed maturity

The FRS [3] data gives in successive working days the daily average interest rate  $I(day)$  between banks. They are given in Table (1) for the eight following maturities

$$\{[m]\} = \{[1], [2], [3], [5], [7], [10], [20], [30]\} \text{ in years .} \quad (49)$$

For many of these maturities, the data extends (10698 events) without break from 1962 to today (we have chosen November 4 as our last event). For a few maturities, the data is restricted to one or more sub-periods within the 1962-today period. The total number of events for each maturity is given in table (1). In particular, it should be stressed that the data for  $[m] = 20$  years consist of two disjoint periods. We have not been able to discover if, during these two periods, the way of computing the daily averages are coherent. We have chosen to group the two sets but we will be led to caution comments.

The interest rates are given in percent by a number with exactly two decimal figures, i.e. by a integer number in basis points. It allows the definition of the following meaningful distributions for a lag  $L$  of one day. These distributions have to be compared to the corresponding theoretical distributions.

### The empirical $N(v, I)$ and $\overline{N}(v, i)$ density distributions

For  $v$  and  $I$  both expressed in basis points and for each of the allowed maturities, the empirical distribution  $N(v, I)$  is the number of occurrences when the interest rate of the FRS on some day was  $I$  and the next day  $I + v$ . These

numbers are statistically very small. It is useful to consider the compounded empirical distribution  $\overline{N}(v, i)$  where  $v$  is still expressed in basis points but the interest  $i$  is expressed in percent by grouping the days when the interest rate is almost  $i$

More precisely, for the data, the  $N(v, I)$  ( $v$  and  $I$  in basis points) density distribution is defined as follows

$$N(v, I) = \text{number of days when } I(\text{day}) = I \text{ and } I(\text{day} + 1) = I + v . \quad (50)$$

Note that, in order to obtain  $I(\text{day}+1)$ , non working days are simply discarded. The average discretised  $\overline{N}(v, i)$  ( $v$  in basis points and  $i$  in percent) density distribution is precisely defined by

$$\overline{N}(v, i) = \sum_{I=100i-49}^{100i+50} N(v, I) . \quad (51)$$

It is the number of days when the interest rate was between  $i - 1/2$  percent ( exactly  $100i - 49$  basis points) and  $i + 1/2$  percent (exactly  $100i + 50$  basis points) and the interest rate has moved by an amount  $v$  basis points by the next day.

### The empirical $w(I)$ and $\overline{w}(i)$ interest distributions

For each allowed maturity, the empirical interest distribution  $w(I)$  is defined as

$$w(I) = \sum_{v_{min}}^{v_{max}} N(v, I) \quad (52)$$

where  $I$  is in basis point. It is the number of days when the interest rate was  $I$  in basis point. The average discrete version  $\overline{w}(i)$  of (52) is defined by

$$\overline{w}(i) = \sum_{I=100i-49}^{100i+50} w(I) . \quad (53)$$

where  $i$  is an integer giving the interest rate in percent. it is simply the number of days when the interest rate was between  $(i - 1/2)$  and  $(i + 1/2)$  percents. The total number of events  $\overline{w} = \sum_i \overline{w}(i)$  depends on the maturity and is given in Table (1).

The  $\overline{w}(i)$  are given in Table (2) for all maturities of  $[m]$  years. In preceding papers it was realized that bins with less than about one thousand events



| $i$                      | 1   | 2   | 3   | 4    | 5    | 6    | 7    | 8    | 9   |
|--------------------------|-----|-----|-----|------|------|------|------|------|-----|
| $\overline{w}(i)_{[1]}$  | 356 | 402 | 664 | 1269 | 1674 | 2147 | 1110 | 1147 | 582 |
| $\overline{w}(i)_{[2]}$  | 60  | 418 | 293 | 569  | 599  | 1530 | 759  | 835  | 561 |
| $\overline{w}(i)_{[3]}$  | 9   | 369 | 524 | 1246 | 1254 | 2166 | 1642 | 1334 | 732 |
| $\overline{w}(i)_{[5]}$  | 0   | 33  | 45  | 130  | 1278 | 2010 | 1791 | 1563 | 778 |
| $\overline{w}(i)_{[7]}$  | 0   | 0   | 154 | 495  | 693  | 1705 | 1775 | 1582 | 880 |
| $\overline{w}(i)_{[10]}$ | 0   | 0   | 28  | 1485 | 1246 | 1949 | 1874 | 1629 | 929 |
| $\overline{w}(i)_{[20]}$ | 0   | 0   | 0   | 1020 | 1320 | 1974 | 1389 | 1341 | 396 |
| $\overline{w}(i)_{[30]}$ | 0   | 0   | 0   | 0    | 289  | 1076 | 968  | 1358 | 981 |

Table 2: The empirical  $\overline{w}(i)_{[m]}$  for  $i = 1 - 9$  and all maturities  $[m]$ .

present too much statistical fluctuations. Hence it is useful to group the data in bins. We have chosen this bin definition identical for each of the allowed maturities. In Table (4), the  $\overline{w}(i_{\text{bin}})$  are given, as an example, for maturity  $[m] = 1$ , together with a definition of the bins.

### The empirical $\overline{N}_{\text{data}}(v, i_{\text{bin}})$ density distributions

The final empirical density distribution  $\overline{N}(v, i_{\text{bin}})$  which enters the  $\chi$ -squared merit function is taken as

$$\overline{N}_{\text{data}}(v, i_{\text{bin}}) = \sum_{i \in i_{\text{bin}}} \overline{w}(i) \overline{N}(v, i) \quad (54)$$

## 6 The definition of $N_{\text{theory}}$ and the $\chi$ -squared merit function

For each maturity, the empirical distribution  $\overline{N}_{\text{data}}(v, i)$  has to be compared to the effective distribution  $\overline{N}_{\text{theory}}(v, i)$  computed from the theoretical distribution evaluated at lag  $L = 1$  day, which means for  $l = N_L$ . This distribution  $\overline{p}(N_L, v, i)$  is computed from the seed by  $N_L$  successive convolutions.

We have

$$\begin{aligned} \overline{N}_{\text{theory}}(v, i) &= \overline{w}(i) \overline{p}(N_L, v, i) \\ \overline{N}_{\text{theory}}(v, i_{\text{bin}}) &= \sum_{i \in i_{\text{bin}}} \overline{N}_{\text{theory}}(v, i) . \end{aligned} \quad (55)$$

|                          |     |     |     |     |     |     |    |    |    |
|--------------------------|-----|-----|-----|-----|-----|-----|----|----|----|
| $i$                      | 10  | 11  | 12  | 13  | 14  | 15  | 16 | 17 | 18 |
| $\overline{w}(i)_{[1]}$  | 469 | 176 | 202 | 105 | 160 | 131 | 65 | 38 | 0  |
| $\overline{w}(i)_{[2]}$  | 445 | 294 | 196 | 157 | 190 | 126 | 55 | 18 | 0  |
| $\overline{w}(i)_{[3]}$  | 333 | 351 | 179 | 173 | 234 | 88  | 59 | 4  | 0  |
| $\overline{w}(i)_{[5]}$  | 341 | 338 | 222 | 216 | 240 | 94  | 39 | 0  | 0  |
| $\overline{w}(i)_{[7]}$  | 332 | 306 | 300 | 250 | 233 | 96  | 26 | 0  | 0  |
| $\overline{w}(i)_{[10]}$ | 306 | 326 | 331 | 258 | 229 | 97  | 10 | 0  | 0  |
| $\overline{w}(i)_{[20]}$ | 204 | 384 | 438 | 251 | 211 | 74  | 5  | 0  | 0  |
| $\overline{w}(i)_{[30]}$ | 282 | 364 | 413 | 295 | 177 | 39  | 0  | 0  | 0  |

Table 3: The empirical  $\overline{w}(i)_{[m]}$  for  $i = 10 - 18$  and all maturities  $[m]$ .

|                                      |      |      |      |      |      |      |      |       |
|--------------------------------------|------|------|------|------|------|------|------|-------|
| $i_{\text{bin}}$                     | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8     |
| $i \in i_{\text{bin}}$               | 1-3  | 4    | 5    | 6    | 7    | 8    | 9-10 | 11-17 |
| $\overline{w}(i_{\text{bin}})_{[1]}$ | 1165 | 1269 | 1674 | 2147 | 1110 | 1147 | 1051 | 877   |

Table 4: Example of the empirical  $\overline{w}(i_{\text{bin}})$  for the maturity  $[m] = [1]$ . The other  $[m]$  can be computed from Table (3).

### The $\chi$ -squared merit function

For each maturity  $[m]$ , the following  $\chi$ -squared function is defined

$$\chi_{[m]}^2 = \sum_{i_{\text{bin}}=1}^8 \sum_{v=-300}^{300} \frac{(\overline{N}_{\text{theory}}(v, i_{\text{bin}}) - \overline{N}_{\text{data}}(v, i_{\text{bin}}))^2}{\sigma(v, i_{\text{bin}})^2} . \quad (56)$$

In this formula, the numerator is the square of the difference between the empirical distribution (51) and the theoretical distribution (55) for a given maturity.. It is a measure of the discrepancy between theory and experiment. The denominator is the square of the modified error  $\sigma(v, i_{\text{bin}})$  According to the previous discussion around (48), this  $\sigma(v, i_{\text{bin}})$  is taken as

$$\begin{aligned} \sigma(v, i_{\text{bin}})_{\text{modified}} &= (N_{\text{data}}(v, i_{\text{bin}})N_{\text{theory}}(v, i_{\text{bin}}))^{1/4} & \text{if } N_{\text{data}}(v, i_{\text{bin}}) \neq 0 \\ \sigma(v, i_{\text{bin}})_{\text{modified}} &= (N_{\text{theory}}(v, i_{\text{bin}}))^{1/4} & \text{if } N_{\text{data}}(v, i_{\text{bin}}) = 0 \end{aligned} \quad (57)$$

With our definitions, the bins where the number of events is zero do not contribute to the  $\chi$ -squared and hence can be removed or kept without altering the results of the minimisations.

Finally, let us note that the limits on the summation over  $v$ ,  $[-300, 300]$ , have been chosen at the point where the distributions in  $v$  have become empirically completely negligible. For example, increasing the limits from 300 to 400 basis points does not change the values of the parameters provided by the minimization method in any appreciable way.

## 7 Seed Parametrisation

In preceding articles [1], [2], we have shown that the FRS distributions can be fitted very closely by using Padé Approximates [0, 4] whose coefficients follow rather simple scaling laws (see (94)).

### 7.1 The form of the Seed. Determination of the parameters

Extrapolating the scaling laws discovered in [1], [2] for lags equal to an integer number of days to values of the lag small compared to one day, as is explained in Appendix (D), one is led to state the following form for the discretized seed

$$s_L S(v_{SV}, i_{SI}) = \frac{1}{s_V} \frac{\alpha(i)}{1 + \gamma(i)} \frac{1}{|v|^{d(i)}} \quad \text{for } v \neq 0 . \quad (58)$$

It depends on three independent parameters  $\alpha$ ,  $\gamma$  and  $d$  which themselves depend on the initial interest rate  $i$  and on the maturity  $[m]$ . In this equation, using our choices of  $s_L$  (35),  $s_V$  (37) and  $s_I$  (39), we see that  $v$  and  $i$  are integers, that the variation of interest rate  $V = s_V v$  has the dimension of basis point, that the initial interest rate  $I = s_I i$  has the dimension of percent, that the contribution of  $s_L S$  to the probability distribution has the dimension of the inverse basis point and that  $\alpha$ ,  $\beta$  and  $d$  have no dimension.

It is first convenient to define  $V_{transition}$  as the variation of interest rate  $V$  at the place where the transition between a constant behavior (in  $v$ ) and the power decrease of the seed occurs. More precisely, it is defined as the value for which the second term in the denominator of (58) becomes equal to the first term in this denominator. Thus when the second term becomes 1. It is linked to the parameter  $\gamma$  by

$$v_{transition} \approx \gamma^{-\frac{1}{d}} . \quad (59)$$

Using the definition (36), we see that this corresponds to a value of the interest rate variation  $V_{transition}$  expressed in basis point of

$$V_{transition} = v_{transition} s_V . \quad (60)$$

It is then easy to understand the meaning of the three parameters:

- The parameter  $\alpha$  is linked, in first approximation, to the constant behaviour of the seed for (non-zero) variations of the interest rate appreciatively smaller than the transition value defined in (59).
- The parameter  $\gamma$  is linked, in first approximation, to the transition value of the variation of the interest rate (59) between a constant behaviour (for small  $v$ ) and a power decrease of the seed for large  $v$ .
- The parameter parameter  $d$  is linked to the asymptotic power decrease of the seed when the variation of the interest rate is large compared to  $v_{transition}$ .

Guided by the findings of [1], [2], we have taken the following dependence of the three parameters in terms of the initial interest rate:

$$\begin{aligned} \alpha(i) &= \alpha_1 e^{\alpha_2(i-i_0)} \\ \gamma(i) &= \gamma_1 e^{\gamma_2(i-i_0)} \\ d(i) &= d_1 e^{d_2(i-i_0)+d_3(i-i_0)^2} . \end{aligned} \quad (61)$$

Again  $i$  is in percent and  $i_0$  is arbitrarily but very conveniently chosen to be 6.5 percents (see the discussion in Appendix (D)). At this point, there are thus seven parameters which are maturity dependent.

## 7.2 Determination of the parameters

To calibrate the model, we have then proceeded along the following steps.

- We have first computed, for each maturity  $[m]$ , the values of the seven parameters  $(\alpha_1, \alpha_2, \gamma_1, \gamma_2, d_1, d_2, d_3)$  which minimize the corresponding  $\chi_{[m]}^2$ .
- Looking at these minimal values, we have discovered that linear extrapolation in  $[m]$  are approximately at work. More precisely, the following parametrisation with straight lines is suggested

$$\begin{aligned}
\alpha_1([m]) &= (\alpha_{11} + \alpha_{12}([m] - [m]_0)) 10^{-5} \\
\alpha_2([m]) &= (\alpha_{21} + \alpha_{22}([m] - [m]_0)) 10^{-5} \\
\gamma_1([m]) &= (\gamma_{11} + \gamma_{12}([m] - [m]_0)) 10^{-6} \\
\gamma_2([m]) &= (\gamma_{21} + \gamma_{22}([m] - [m]_0)) 10^{-1} \\
d_1([m]) &= (d_{11} + d_{12}([m] - [m]_0)) \\
d_2([m]) &= (d_{21} + d_{22}([m] - [m]_0)) 10^{-3} \\
d_3([m]) &= (d_{31} + d_{32}([m] - [m]_0)) 10^{-4} .
\end{aligned}$$

The  $[m]_0$  which appears in these formulae is arbitrarily chosen to be [15] (years) close to the middle of the interval  $[[m] = [1], [m] = [30]]$ . The normalisation  $10^{-5}, 10^{-6}, \dots$  have been chosen conveniently. Clearly, the parameters with a 2 as their second index are related to the slopes of the straight lines while the parameters with a 1 as their first index are the ordinates of the lines for  $[m] = [m]_0$ . Altogether, there are now fourteen constant parameters.

- To determine the empirical values of the fourteen parameters we have chosen to minimize the total merit function

$$\chi_{total}^2 = \sum_{[m] \in \{[m]\}} \chi_{[m]}^2 \quad (62)$$

which is simply the sum of the  $\chi$ -squared for the eight maturities (49). See the discussion in Appendix (D) where the results are summarized in Tables (10)-(11).

- It turns out that only eight of the fourteen parameters are really relevant and that six of them, namely  $\alpha_{21}, \alpha_{22}, d_{12}, d_{21}, d_{22}, d_{31}$  can be safely put to zero (see the discussion in Appendix (D))

$$\alpha_{21} = \alpha_{22} = d_{12} = d_{21} = d_{22} = d_{31} = 0 . \quad (63)$$

|       | $\alpha_{11}$ | $\alpha_{12}$ | $\gamma_{11}$ | $\gamma_{12}$ | $\gamma_{21}$ | $\gamma_{22}$ | $d_{11}$    | $d_{32}$    |
|-------|---------------|---------------|---------------|---------------|---------------|---------------|-------------|-------------|
| value | 2.39          | 0.018         | 5.78          | 0.20          | -2.90         | -0.100        | 3.030       | 0.909       |
| C.I.  | $\pm 0.08$    | $\pm 0.004$   | $\pm 0.99$    | $\pm 0.04$    | $\pm 0.49$    | $\pm 0.032$   | $\pm 0.024$ | $\pm 0.067$ |

Table 5: The selected  $\alpha$ ,  $\gamma$ ,  $d$  parameters with the choice  $i_0 = 6.5$ ,  $[m]_0 = 15$  and  $\alpha_{21} = \alpha_{21} = d_{12} = d_{21} = d_{22} = d_{31} = 0$ . The C.I. are the confidence intervals.

| $[m]$                | [1]  | [2]  | [3]  | [5]  | [7]  | [10] | [20] | [30] |
|----------------------|------|------|------|------|------|------|------|------|
| $\alpha \times 10^5$ | 2.14 | 2.15 | 2.17 | 2.21 | 2.24 | 2.30 | 2.4  | 2.6  |

Table 6: The values of  $\alpha \times 10^5$  as a function of  $[m]$ .

- A last minimisation, taking again  $\chi_{total}^2$  as the merit function and taking into account the restrictions (63), has produced the final values of the eight remaining non-zero parameters as given in Table (5).

### 7.3 Comparison with the data. Discussion

We are now in a position for a detailed comparison of the results of our fits with the data. The values obtained for the parameters are summarized in Table (6) for  $\alpha$ , in Table (7) for  $\gamma$ , in Table (8) for  $d$  and in Table (9) for  $v_{transition}$ . They are given in natural units (see [1] for a precise discussion of these units).

The parameter  $\alpha$

The parameter  $\alpha$  does not depend of the initial interest rate and is a slowly increasing function of the maturity  $[m]$ . Its values (multiplied by  $10^5$ ) are expressed in inverse basis points and given in Table (6).

The parameter  $\gamma$

The parameter  $\gamma$  depends both on the maturity and on the interest rate. Its

| $\gamma \times 10^6$ | i=1  | i=3  | i=5  | i=7  | i=9  | i=11 | i=13  |
|----------------------|------|------|------|------|------|------|-------|
| [m]=[1]              | 6.85 | 5.08 | 3.77 | 2.8  | 2.07 | 1.54 | 1.14  |
| [m]=[2]              | 7.72 | 5.61 | 4.08 | 2.96 | 2.15 | 1.57 | 1.14  |
| [m]=[3]              | 8.66 | 6.17 | 4.39 | 3.13 | 2.23 | 1.59 | 1.13  |
| [m]=[5]              | 10.8 | 7.38 | 5.05 | 3.46 | 2.37 | 1.62 | 1.11  |
| [m]=[7]              | 13.3 | 8.74 | 5.75 | 3.78 | 2.49 | 1.63 | 1.07  |
| [m]=[10]             | 17.9 | 11.1 | 6.86 | 4.25 | 2.63 | 1.63 | 1.01  |
| [m]=[20]             | 43.9 | 22.3 | 11.3 | 5.71 | 2.89 | 1.46 | 0.742 |
| [m]=[30]             | 98.6 | 40.9 | 16.9 | 7.02 | 2.91 | 1.2  | 0.499 |

Table 7: The values of  $\gamma \times 10^6$  as a function of  $[m]$  and  $i$ .

dimension is the inverse of the basis point raised to the power  $1/d$ . Its is given in Table (7) multiplied by  $10^6$ . In fact  $v_{transition}$  defined before, directly derived from  $\gamma$  and given in Table (9), carries a more intuitive meaning.

The parameter  $d$

The parameter  $d$  (which is a pure number) depends on the maturity and on the initial rate. It is given in Table (8). It should be remarked that it is empirically always close to 3. One may even wonder if the exact value of 3 would not be a critical exponent of the problem and have general validity.

The transition point  $v_{transition}$

With the values of the parameters in (5), the values of the transition points  $v_{transition}$  are given in Table (9) for the odd  $i$ 's and all the maturities. One sees that, as expected by intuitive arguments,  $v_{transition}$  is an increasing function of the initial interest rate and a decreasing function of the maturity.

The  $\chi$ -squared for the best parameters

The total  $\chi$ -squared as well as the  $\chi$ -squared for all maturities can be found in table (12). In general, as discussed in the appendix, the reduced  $\chi$ -squared are

| $d$        | i=1  | i=3  | i=5  | i=7  | i=9  | i=11 | i=13 |
|------------|------|------|------|------|------|------|------|
| $[m]=[1]$  | 2.92 | 2.98 | 3.02 | 3.03 | 3.01 | 2.95 | 2.87 |
| $[m]=[2]$  | 2.92 | 2.99 | 3.02 | 3.03 | 3.01 | 2.96 | 2.88 |
| $[m]=[3]$  | 2.93 | 2.99 | 3.02 | 3.03 | 3.01 | 2.96 | 2.89 |
| $[m]=[5]$  | 2.95 | 3.0  | 3.02 | 3.03 | 3.01 | 2.98 | 2.92 |
| $[m]=[7]$  | 2.96 | 3.0  | 3.03 | 3.03 | 3.02 | 2.99 | 2.94 |
| $[m]=[10]$ | 2.99 | 3.01 | 3.03 | 3.03 | 3.02 | 3.0  | 2.97 |
| $[m]=[20]$ | 3.07 | 3.05 | 3.03 | 3.03 | 3.04 | 3.06 | 3.09 |
| $[m]=[30]$ | 3.16 | 3.08 | 3.04 | 3.03 | 3.06 | 3.12 | 3.21 |

Table 8: The values of  $d$  as a function of  $[m]$  and  $i$ .

| $v_{transition}$ | i=1 | i=3 | i=5 | i=7 | i=9 | i=11 | i=13 |
|------------------|-----|-----|-----|-----|-----|------|------|
| $[m]=[1]$        | 59  | 60  | 62  | 68  | 78  | 93   | 117  |
| $[m]=[2]$        | 56  | 57  | 61  | 67  | 77  | 92   | 115  |
| $[m]=[3]$        | 53  | 55  | 59  | 66  | 76  | 90   | 113  |
| $[m]=[5]$        | 48  | 52  | 56  | 64  | 74  | 88   | 110  |
| $[m]=[7]$        | 44  | 48  | 54  | 62  | 72  | 87   | 107  |
| $[m]=[10]$       | 39  | 44  | 51  | 59  | 70  | 85   | 104  |
| $[m]=[20]$       | 26  | 34  | 43  | 54  | 67  | 81   | 97   |
| $[m]=[30]$       | 19  | 27  | 37  | 50  | 65  | 80   | 92   |

Table 9: The values of  $v_{transition}$  (see (59),(??)), as a function of  $[m]$  and  $i$ .



close to one and hence very good, except for  $[m] = 20$  where there are some uncertainties in the data and to a lesser extend for  $[m] = 5$ . This is reflected in the quality of the agreement between the empirical distributions and the data as seen in the plots below.

## 7.4 A few plots

All the plots, for lag one and higher, for all bins and for all maturities show good to very good agreement between our theoretical curves and the data provided by the FRS site [3]. Both the central region where the variation  $v$  of the interest rates is around zero and the tails where  $v$  becomes large are very well accounted for. We have selected randomly eleven plots (1)-(11) as examples of the fits. In particular, the domains of the selected plots are given in detail in the figure captions.

## 8 Conclusions

In this paper, a microscopic theory of the term structure of interest rates has been developed. Convolution techniques, implying about ten successive convolutions, combined with time translation invariance lead to a time scaled theory where the term structure for practical lags (one day or more) can be deduced from a seed function living at a relatively small lag scale (a few seconds).

The previously discovered scaling laws, which were found to be valid at lags expressed in days, suggest forms for the possible seeds and imply a discretisation of the problem.

Our new results show that the scaling law assumptions are even simpler at the microscopic lag scale. Indeed, it is shown that the FRS data are amazingly well reproduced (for a lag of one day or two days but the results easily extend to higher lags) by assuming that the seed has the critical form of a self-organized [9] econo-physical system. In other words, a very simple power law behavior emerges with essentially only one  $v^{-d}$  term. The exponent  $d$  is of the order of three in close agreement with the tail behaviours obtained previously using the Hill [10] estimator.

These results open the door to two major issues, one rather theoretical and the other more practical and pragmatic.

Since the seed has such a simple scaling form, it suggests the existence of an underlying statistical model. The discovery of the basic ingredients and laws leading in a natural and systematic way to the scaling would be a major achievement which, if attained, may also lead to new theoretical insights in related but different contexts.

We expect that a self-organized structure may be at work, with  $d = 3$ , or close to three, as critical exponent for all the maturities. We hope to come back to this issue.

Finally, one may wonder how this theory and its predictions can be efficiently used in the context of risk management (e.g. Value-at-Risk computation) and fixed-income derivative pricing. For the time being, studies are under way to measure the add-value of using this model instead of the traditional ones. First results will be provided in a near future.

## A A naïve introduction to Dirac type distributions

### A.1 Formal definition

A distribution  $G(V)$  is a continuous linear functional on some space of functions of a real variable  $V$ . The Dirac  $\delta(V)$  distribution, which has compact support, is defined on a continuous  $C^\infty$  function  $f(V)$  at  $V = 0$  by

$$(\delta(V), f(V)) = f(0) . \quad (64)$$

The  $j$ th derivative  $G^{[j]}$  of a distribution of compact support  $G(V)$  is defined as

$$(G^{[j]}(V), f(V)) = (-1)^j (G(V), f^{[j]}(V)) \quad (65)$$

where

$$f^{[j]}(V) = \frac{d^j f(V)}{dV^j} \quad (66)$$

is the  $j$ th derivative of the function  $f(V)$ .

In particular, the second derivative of the Dirac distribution

$$(\delta^{[2]}(V), f(V)) = f^{[2]}(0) \quad (67)$$

is the second derivative of  $f(V)$  evaluated at  $V = 0$ .

Without being mathematically rigorous, these definitions can be seen in a more intuitive way. As Laurent Schwartz, the inventor of the general distribution concept, used to caution : The formulae given below, with physicist notations, have to be used with great care and have to be justified in a detailed and precise way. But he also recognized that they were very convenient to guess properties of distributions.

### A.2 Discussion and approximations for the Dirac distributions

The distribution  $\delta(V)$  is a generalized function which can be thought as being zero for  $V < 0$  and  $V > 0$  and infinite at  $V = 0$  in such a way that, in formal agreement with (64)

$$\int_{-\infty}^{+\infty} \delta(V) f(V) dV = f(0) . \quad (68)$$

There are many functions which can approximate the delta function. Let us give two.

1. The step-type function

$$\delta(V) = \lim_{s \rightarrow 0} D(V, s) \begin{cases} D(V, s) = 0 & \text{for } V < -\frac{s}{2} \\ D(V, s) = \frac{1}{s} & \text{for } -\frac{s}{2} \leq V \leq \frac{s}{2} \\ D(V, s) = 0 & \text{for } \frac{s}{2} < V \end{cases} \quad (69)$$

has compact support. It is not difficult do the Riemann integration and prove that for some  $\hat{V}$

$$-\frac{s}{2} \leq \hat{V} \leq \frac{s}{2} \quad (70)$$

one has

$$\begin{aligned} \int_{-\infty}^{+\infty} D(V, s) f(V) dV &= f(\hat{V}) \left( \int_{-\frac{s}{2}}^{+\frac{s}{2}} D(V, s) dV \right) \\ &= f(\hat{V}) . \end{aligned} \quad (71)$$

The limit for  $s \rightarrow 0$  reproduces clearly (68) as  $f(\hat{V}) \rightarrow f(0)$ .

2. The Dirac distribution can also be approximated by the continuous function

$$\begin{aligned} \delta(V) &= \lim_{\Delta \rightarrow 0} E(V, \Delta) \\ E(V, \Delta) &= \frac{\Delta}{\pi (\Delta^2 + V^2)} . \end{aligned} \quad (72)$$

This form is very interesting, as it shows that the  $\delta(V)$  function is the boundary value of a continuous function of two variables. For positive  $\Delta$ , this function is purely positive for all  $V$ .

A step-type approximation for the second derivative of the delta function is given as follows. Consider the intervals  $A = [-3s/2, -s/2]$ ,  $B = [-s/2, +s/2]$ ,  $C = [+s/2, +3s/2]$ , the step function which is zero outside the three intervals, has value  $1/s$  in the intervals  $A$  and  $C$  and  $-2/s$  in the interval  $B$  and let  $s \rightarrow 0$ .

## B Formal discussion of the Master Equation. Continuous variables

In the following it is shown that the formal solution of the master equation is fully compatible with the convolution equation (18) whatever be the intermediate time

Suppose that, the probability distribution has a Taylor power expansion, around  $L = 0$ , of the form

$$p(L, V, I_0) = \sum_{n=0}^{\infty} \frac{L^n}{n!} p^{(n)}(V, I_0) . \quad (73)$$

where  $p^{(n)}(V, I_0)$  is the  $n$ 'th order derivative of  $p$  with respect to  $L$  evaluated at  $L = 0$

$$p^{(n)}(V, I_0) = \partial_L^n p(L, V, I_0) \Big|_{L=0} . \quad (74)$$

The zeroth order  $p^{(0)}(V, I_0)$  is known by Eq.(21) and the first order  $p^{(1)}(V, I_0)$  by Eq.(24)

$$\begin{aligned} p^{(0)}(V, I_0) &= p(0, V, I_0) = \delta(V) \\ p^{(1)}(V, I_0) &= S(V, I_0) . \end{aligned} \quad (75)$$

Introducing this power expansion in the master equation (27) and equating, in the left and right hand sides, the coefficients of the same powers in  $L$ , one gets the recurrence equations

$$p^{(n)}(I_f - I_0, I_0) = \int_{-\infty}^{+\infty} p^{(n-1)}(I_f - I_i, I_i) S(I_i - I_0, I_0) dI_i , \quad n = 1, \dots, \infty . \quad (76)$$

It is easily seen that the equation for  $n = 1$  is an identity. For  $n = 2$ , one finds

$$p^{(2)}(I_f - I_0, I_0) = \int_{-\infty}^{+\infty} S(I_f - I_1, I_1) S(I_1 - I_0, I_0) dI_1 \quad (77)$$

where the dummy integration variable has been called  $I_1$ . For arbitrary  $n \geq 1$ , one obtains

$$\begin{aligned} p^{(n+1)}(I_f - I_0, I_0) &= \int_{-\infty}^{+\infty} dI_n \int_{-\infty}^{+\infty} dI_{n-1} \dots \int_{-\infty}^{+\infty} dI_1 \\ &\quad S(I_f - I_n, I_n) S(I_n - I_{n-1}) \dots S(I_1 - I_0, I_0) \\ &= \int_{-\infty}^{+\infty} \left( \prod_{p=1}^n dI_p \right) \left( \prod_{q=0}^n S(I_{q+1} - I_q, I_q) \right) , \quad I_{n+1} = I_f . \end{aligned} \quad (78)$$

It is not difficult to check that the formal power expansion (73), (75), (77), (78) satisfies the convolution equation (18) without any further condition. Hence, it can reasonably be supposed that one can focus on the master equation (27) together with its natural boundary conditions.

## C Master Equation with a simplified Scaling Law and no $I_0$ dependence. The Gauss distribution solution

As suggested in section (3), it is convenient to study analytically the solution of the master equation with the seed (29)

$$S(V, I_0) = \kappa(I_0) \partial_V^2 \delta(V) . \quad (79)$$

In order to simplify the problem let us limit ourselves to suppose that the distribution  $p$  does not depend on  $I_0$

$$p(L, V, I_0) = \hat{p}(L, V) \quad (80)$$

as well as the seed (29) and hence

$$\kappa(I_0) = \hat{\kappa} . \quad (81)$$

Eq.(30) becomes

$$\partial_L \hat{p}(L, V) = \hat{\kappa} \partial_V^2 \hat{p}(L, V) . \quad (82)$$

To simplify the problem even further, “scaling laws” which are approximately but not exactly verified by the data [2], are assumed. It has to be emphasized that the data do not support exactly these scaling laws. Hence the results we will obtain in this section cannot be hoped to be correct. On the other hand, it is worth to quote them since they have as a consequence the very common but wrong belief that, in a natural way, distributions should fall off at large  $V$  as exponentials.

The ultra simplification of the empirical results found in [1], [2] is equivalent to the statement that the distributions depend on the reduced variable  $\tilde{V}$  (84) and scales (83) in the natural way as  $1/\sqrt{L}$

$$p(L, V, I_0) = \frac{\tilde{p}(\tilde{V})}{\sqrt{L}} \quad (83)$$

$$\tilde{V} = \frac{V}{\sqrt{L}} . \quad (84)$$

Replacing  $p$  by its guess (83) in (30), one finds the final equation

$$2\kappa \partial_{\tilde{V}}^2 \tilde{p}(\tilde{V}) + \partial_{\tilde{V}} \left( \tilde{V} \tilde{p}(\tilde{V}) \right) = 0 . \quad (85)$$

The solutions of this equation can easily be found. Indeed, it can be written

$$\partial_{\tilde{V}} \left( 2\kappa \partial_{\tilde{V}} \tilde{p}(\tilde{V}) + \tilde{V} \tilde{p}(\tilde{V}) \right) = 0 \quad (86)$$

whose general solution is

$$2\kappa \partial_{\tilde{V}} \tilde{p}(\tilde{V}) + \tilde{V} \tilde{p}(\tilde{V}) = R_1 \quad (87)$$

where  $R_1$  is an arbitrary constant. To solve this first order differential equations, the homogeneous equation is solved

$$2\kappa \partial_{\tilde{V}} \tilde{p}^h(\tilde{V}) + \tilde{V} \tilde{p}^h(\tilde{V}) = 0 \quad (88)$$

giving the homogeneous solution  $\tilde{p}^h(\tilde{V})$

$$\tilde{p}^h(\tilde{V}) = H e^{-\frac{\tilde{V}^2}{4\kappa}}. \quad (89)$$

Assuming (the method of variation of constants) that  $H$  depends on  $\tilde{V}$ , introduce (89) in (87) to obtain the equation

$$\partial_{\tilde{V}} H(\tilde{V}) = \frac{R_1}{2\kappa} e^{\frac{\tilde{V}^2}{4\kappa}} \quad (90)$$

which can be solved by a simple integration

$$H(\tilde{V}) = \frac{R_1}{2\kappa} \int_0^{\tilde{V}} dx e^{\frac{x^2}{4\kappa}} + R_2 \quad (91)$$

where  $R_2$  is the second constant of integration. One has now to plot the final solution obtained by replacing the solution for  $H$  (91) in the homogeneous solution (89)

$$\begin{aligned} \tilde{p}(\tilde{V}) &= H(\tilde{V}) e^{-\frac{\tilde{V}^2}{4\kappa}} \\ &= \left( \frac{R_1}{2\kappa} \int_0^{\tilde{V}} dx e^{\frac{x^2}{4\kappa}} + R_2 \right) e^{-\frac{\tilde{V}^2}{4\kappa}}. \end{aligned} \quad (92)$$

In fact, we are looking for a distribution function  $p(L, V, I_0)$  which is, in first order symmetric in  $V$ , hence for a scaled function  $\tilde{p}(\tilde{V})$  which is also symmetrical in  $\tilde{V}$ . Since by (89) the homogeneous solution  $\tilde{p}^h(\tilde{V})$  is symmetrical, one looks for a symmetrical  $H(\tilde{V})$  solution. This clearly implies that  $R_1$  should be zero.

We thus conclude that the solution of the master equation with strict scaling law leads to the familiar Gauss type behavior

$$\tilde{p}(\tilde{V}) = R_2 e^{-\frac{\tilde{V}^2}{4\kappa}} \quad (93)$$

which is not at all sustained by the data as the main result of our preceding investigation has shown that the asymptotically the term structure decreases as a power of  $1/V^d$  with  $d$  around three to four.

## D Form of the Seed. Phenomenological Discussion

In preceding articles [1], [2], distributions fitting very closely the FRS data have been obtained using Padé Approximants [0, 4] (see (94)) i.e. a polynomial of zero degree in  $v$  in the numerator divided by a polynomial of fourth degree in the denominator

$$p(v, i) = \frac{a(i)}{1 + b(i)v^2 + c(i)v^4} . \quad (94)$$

Moreover, it was shown that the Padé coefficients  $(a(i), b(i), c(i))$ , which also depend on the maturity, follow rather simple scaling laws. Extrapolating these scaling laws for values of the lag small compared to one day have lead us to guess suitable forms for the seed. Though, the asymptotic behavior  $|v|^{-d}$  for large  $v$  with  $d$  equal four which follows from (94) is not incompatible with the data, Hill estimators [10] were pointing towards a somewhat smaller value of  $d$ . Precise values can be found in [1].

Guided by this work and by general ideas about scaling laws and self-organized criticality, we have progressively been led to a simpler educated guesses for the seed, namely (58), where three parameters  $\alpha$ ,  $\gamma$  and  $d$ , in principle, depend on the initial interest rate  $i$  and maturity  $[m]$ . For the dependence in  $i$ , we were led to (61) where  $i_0$  can be arbitrarily chosen. The seven coefficients themselves are thought to be, in first approximation linear in the maturity  $m$ . More precisely, we have thus used (61) and (62) with their convenient normalisations. They depend on initially on fourteen independent constant parameters.

At first, we have chosen, as is generally advisable to diminish the relationships between the relevant parameters, the arbitrary  $i_0$  and  $[m]_0$  in the middle of their respective domains and more precisely  $i_0 = 6$  and  $[m] = [15]$ . Starting with this form of the seed and allowing for  $N_L = 10$  iterations (see the discussion following eq.(34), after minimizing the total  $\chi$ -squared we find the results of Tables (10) and (11) for the fourteen parameters in (62). The values of the resulting  $\chi_{[m]}^2$  and  $\chi_{total}^2$  are given in Table (12) together with the number of degrees of freedom  $N_f$  and the reduced  $\chi$ -squared  $\chi^2/N_f$  for each maturity.

Let us now discuss the results of the Tables (12), (10), (11) and their implications for the choice of the parameters.

The  $\chi^2$

From Table (12) we see that the reduced  $\chi$ -squared are rather good except for  $[m] = [20]$ . Let us recall that there is some incoherences for this maturity



|       | $\alpha_{11}$ | $\alpha_{12}$ | $\alpha_{21}$ | $\alpha_{22}$ | $\gamma_{11}$ | $\gamma_{12}$ | $\gamma_{21}$ | $\gamma_{22}$ |
|-------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| value | 2.46          | 0.022         | 0.005         | -0.015        | 6.53          | 0.230         | -2.90         | -0.119        |
| C.I.  | $\pm 0.08$    | $\pm 0.0042$  | $\pm 1.57$    | $\pm 0.122$   | $\pm 0.99$    | $\pm 0.039$   | $\pm 0.49$    | $\pm 0.032$   |

Table 10: The  $\alpha$  and  $\gamma$  parameters for the initial arbitrary choice of  $i_0 = 6$  and  $[m]_0 = [15]$ . The C.I. are the confidence intervals.

|       | $d_{11}$    | $d_{12}$     | $d_{21}$   | $d_{22}$    | $d_{31}$   | $d_{32}$    |
|-------|-------------|--------------|------------|-------------|------------|-------------|
| value | 3.015       | 0.0001       | 0.04       | 0.005       | -0.01      | 0.907       |
| C.I.  | $\pm 0.024$ | $\pm 0.0005$ | $\pm 2.47$ | $\pm 0.161$ | $\pm 0.82$ | $\pm 0.067$ |

Table 11: The  $d$  parameters for the initial arbitrary choice of  $i_0 = 6$  and  $[m]_0 = [15]$

. The C.I. are the confidence intervals.

|              | [1]  | [2]  | [3]  | [5]  | [7]  | [10] | [20] | [30] | <i>total</i> |
|--------------|------|------|------|------|------|------|------|------|--------------|
| $\chi^2$     | 1064 | 1164 | 930  | 1440 | 1090 | 996  | 1832 | 828  | 9347.44      |
| $N_f$        | 1099 | 931  | 959  | 871  | 853  | 793  | 691  | 651  | 6911         |
| $\chi^2/N_f$ | 0.97 | 1.25 | 0.97 | 1.65 | 1.28 | 1.26 | 2.65 | 1.27 | 1.35         |

Table 12: The  $\chi^2$ , the reduced  $\chi^2/N_f$  and the number of degree of freedom  $N_f$

as the data is composed of two disjointed sets of points corresponding to two disjointed periods in time.

The parameters  $\alpha$

From Table (10), we see that two parameters  $\alpha_{21}$  and  $\alpha_{22}$  are very close to zero with errors much larger than their values. This obviously points to a zero value for the combined parameter  $\alpha_2$  which, we recall, sets the dependence in  $i$  of the parameter  $\alpha$ . It means that whatever be the level of the interest rate, for very short intervals the probability of the occurrence of a certain small variation (before the decrease in  $|v|^{-d}$  plays a role) does not depend on the interest rate. This is an interesting conclusion. From Table (10), we also learn that the two other parameters ( $\alpha_{11}$  and  $\alpha_{12}$ ) are meaningful. We conclude that the parameter  $\alpha$ , which essentially does not depend on the initial interest rate, increases appreciably with the maturity.

The parameters  $\gamma$

The four  $\gamma$  parameters are all significant. This means that  $v_{transition}$  ((59) where the constant behaviour of the seed tends to become a power behaviour is highly dependent on the maturity and on the initial interest rate. As discussed in the main part of the text, this behaviour conforms to our expectations.

The parameters  $d$

From Table (11), we see that of the six  $d$  parameters, four, namely  $d_{12}, d_{22}, d_{22}, d_{31}$ , can safely be put to zero. This means that  $d$  is almost constant for every maturity and every initial interest rate. Except for  $d_{11}$  which as expected is very close to 3, the only parameter which survive is  $d_{32}$ . It follows that  $d$  has a joint behaviour linear in maturity and parabolic in the initial interest rate. Amazingly enough the arbitrary parameters  $i_0 = 6$  and  $[m]_0 = [15]$  which were chosen in the middle of their respective domains were almost optimal choices which are further discussed below.

In order to obtain the best value of  $i_0$  leading to the best adjusted parabola for the dependence of  $d$  in  $i$ , we have rerun our minimization programs putting to zero the six un-significant parameters  $\alpha_{21}, \alpha_{22}, d_{12}, d_{21}, d_{22}, d_{31}$  but allowing the best choice for the arbitrary parameter  $i_0$ . The results are given in Table (5). They show that  $i_0$  around 6.5 would have even be a better choice to make  $d_{12}, d_{21}, d_{22}, d_{31}$  equal to zero. In the main part of the text we have adhered to this choice.

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### Figure Caption

Figure 1. The comparison between the theoretical curve (line) and the empirical curve (crosses) for  $[m] = [1]$  and for the first bin  $i_{bin} = 1$  which contains the initial interest rates  $i = 1, 2, 3$ . The related total number of events is found in (4). The horizontal axis is in basis points. The vertical axis represents the number of events. The size of the arms of the crosses are estimated errors equal approximatively to the square root of the number of events. The curves for the bins  $i_{bin} = 4$  and  $i_{bin} = 8$  and for the same maturity  $[m] = [1]$  are given in Figures (2) and (3) respectively. The analogous curves for the other bins present fits of equivalent quality.

Figure 2.

The comparison between the theoretical curve (line) and the empirical curve (crosses) for  $[m] = [1]$  and for the bin  $i_{bin} = 4$  which contains the initial interest rate  $i = 6$ . The related total number of events is found in (4). The horizontal axis is in basis points. The vertical axis represents the number of events. The size of the arms of the crosses are estimated errors equal approximatively to the square root of the number of events. The curves for the bins  $i_{bin} = 1$  and  $i_{bin} = 8$  and for the same maturity  $[m] = [1]$  are given in Figures (1) and (3) respectively. The analogous curves for the other bins present fits of equivalent quality.

Figure 3.

The comparison between the theoretical curve (line) and the empirical curve (crosses) for  $[m] = [1]$  and for the bin  $i_{bin} = 8$  which contains the initial interest rates  $i = 11 - 17$ . The related total number of events is found in (4). The horizontal axis is in basis points. The vertical axis represents the number of events. The size of the arms of the crosses are estimated errors equal approximatively to the square root of the number of events. The curves for the bins  $i_{bin} = 1$  and  $i_{bin} = 4$  and for the same maturity  $[m] = [1]$  are given in Figures (1) and (2) respectively. The analogous curves for the other bins present fits of equivalent quality.

Figure 4.

The comparison between the theoretical curve (line) and the empirical curve

(crosses) for  $[m] = [3]$  and for the bin  $i_{bin} = 3$  which contains the initial interest rate  $i = 5$ . The related total number of events can be read in Table (2). The horizontal axis is in basis points. The vertical axis represents the number of events. The size of the arms of the crosses are estimated errors equal approximatively to the square root of the number of events. The analogous curves for the same maturity  $[m] = [3]$  for the other bins, as well as those for maturity  $[m] = [2]$ , present fits of equivalent quality.

Figure 5.

The comparison between the theoretical curve (line) compared with the empirical curve (crosses) for  $[m] = [5]$  and for the first bin  $i_{bin} = 4$  which contains the initial interest rate  $i = 6$ . The related total number of events can be read in (2). The horizontal axis is in basis points. The vertical axis represents the number of events. The size of the arms of the crosses are estimated errors equal approximatively to the square root of the number of events. The analogous curves for the same maturity  $[m] = [5]$  for the other bins, as well as those for maturity  $[m] = [7]$ , present fits of equivalent quality.

Figure 6.

The comparison between the theoretical curve (line) compared with the empirical curve (crosses) for  $[m] = [10]$  and for the first bin  $i_{bin} = 5$  which contains the initial interest rate  $i = 7$ . The related total number of events is found in (2). The horizontal axis is in basis points. The vertical axis represents the number of events. The size of the arms of the crosses are estimated errors equal approximatively to the square root of the number of events. The analogous curves for the same maturity  $[m] = [10]$  for the other bins present fits of equivalent quality.

Figure 7.

The comparison between the theoretical curve (line) compared with the empirical curve (crosses) for  $[m] = [20]$  and for the first bin  $i_{bin} = 3$  which contains the initial interest rate  $i = 5$ . The related total number of events is found in (2). The horizontal axis is in basis points. The vertical axis represents the number of events. The size of the arms of the crosses are estimated errors equal approximatively to the square root of the number of events. The analogous curves for the same maturity  $[m] = [20]$  for the other bins present fits of

equivalent quality.

Figure 8.

The comparison between the theoretical curve (line) compared with the empirical curve (crosses) for  $[m] = [30]$  and for the first bin  $i_{bin} = 7$  which contains the initial interest rates  $i = 9 - 10$ . The related total number of events is found in (2), (3). The horizontal axis is in basis points. The vertical axis represents the number of events. The size of the arms of the crosses are estimated errors equal approximatively to the square root of the number of events. The analogous curves for the same maturity  $[m] = [30]$  for the other bins present fits of equivalent quality.

Figure 9.

The comparison between the theoretical curve (line) compared with the empirical curve (crosses) for the tail (i.e. for  $v$  greater than 5 basis points) for  $[m] = [1]$  for the bin  $i_{bin} = 7$  which contains the initial interest rate  $i = 9 - 10$ . The horizontal axis is in basis points. The vertical axis represents the number of events. The size of the arms of the crosses are estimated errors equal approximatively to the square root of the number of events. The analogous curves for the other bins and the other maturities present fits of equivalent quality. The analogous curves for essentially all maturities all bins, when they make sense, present fits of equivalent quality.

Figure 10.

The comparison between the theoretical curve for  $[m] = [5]$  and for a Lag of 2 days (line) compared with the empirical curve (crosses) for the first bin  $i_{bin} = 3$  which contains the initial interest rates  $i = 5$ . The related total number of events is found in (3). The horizontal axis is in basis points. The vertical axis represents the number of events. The size of the arms of the crosses are estimated errors equal approximatively to the square root of the number of events. We have checked that the analogous curves obtained by further convolutions, for all maturities present fits of equivalent quality.

Figure 11.

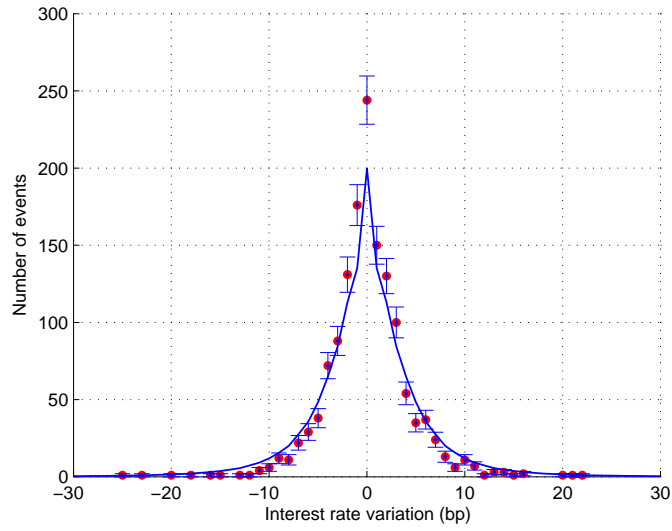
The comparison between the theoretical curve for  $[m] = [1]$  for a Lag of 2 days (line) compared with the empirical curve (crosses) for the tail (i.e. for  $v$  greater

than 5 basis points) for the bin  $i_{bin} = 5$  which contains the initial interest rates  $i = 7$ . The horizontal axis is in basis points. The vertical axis represents the number of events. The size of the arms of the crosses are estimated errors equal approximatively to the square root of the number of events. The analogous curves of the other bins present fits of equivalent quality.



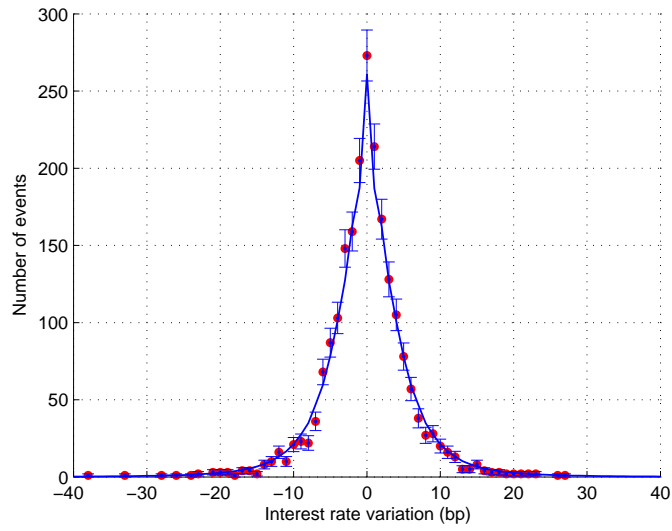
**Figure 1**

Maturity=[1], Initial Interest Rate  $\approx 1-3$ , Lag=1



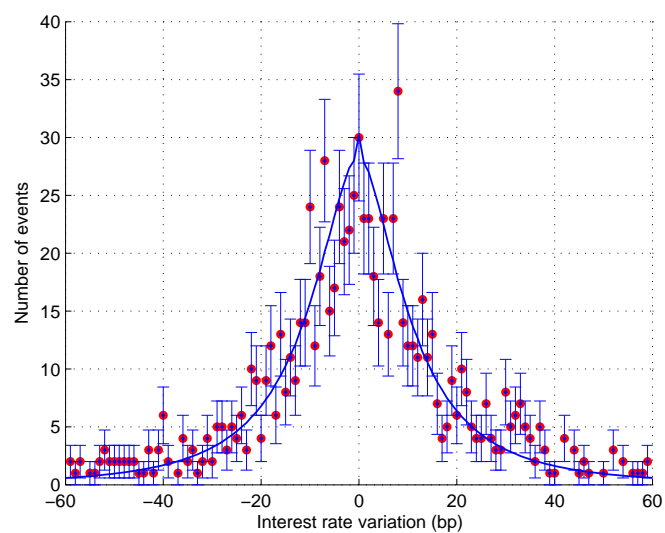
**Figure 2**

Maturity=[1], Initial Interest Rate  $\approx 6$ , Lag=1



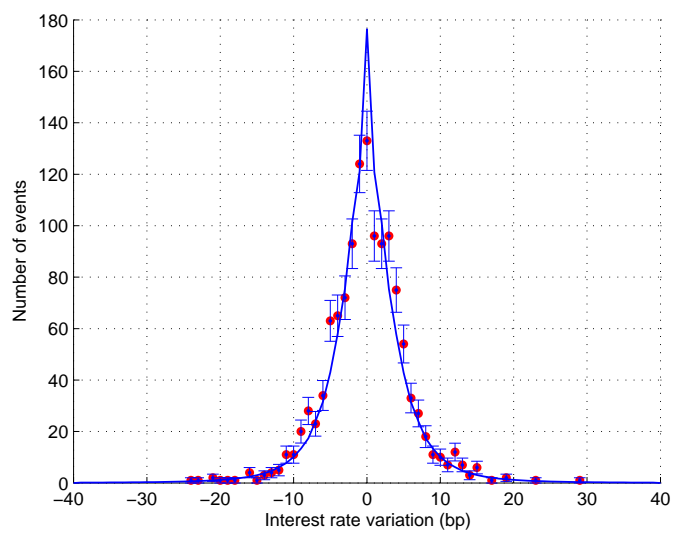
**Figure 3**

Maturity=[1], Initial Interest Rates  $\approx 11-17$ , Lag=1



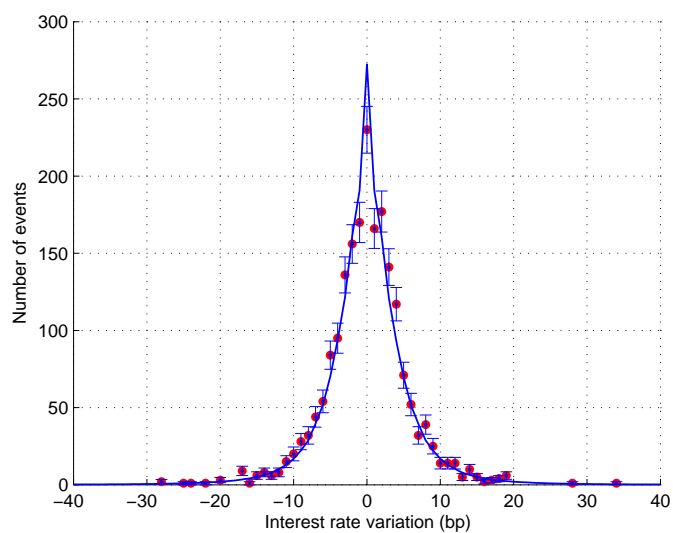
**Figure 4**

Maturity=[3], Initial Interest Rate  $\approx 5$ , Lag=1



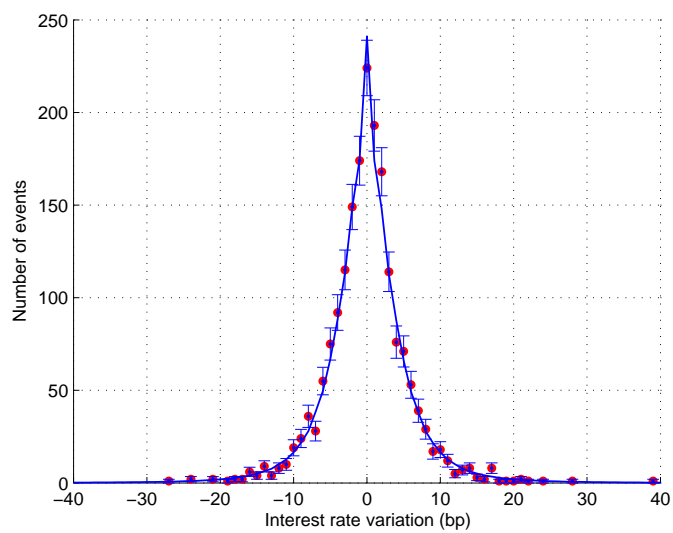
**Figure 5**

Maturity=[5], Initial Interest Rate  $\approx 6$ , Lag=1



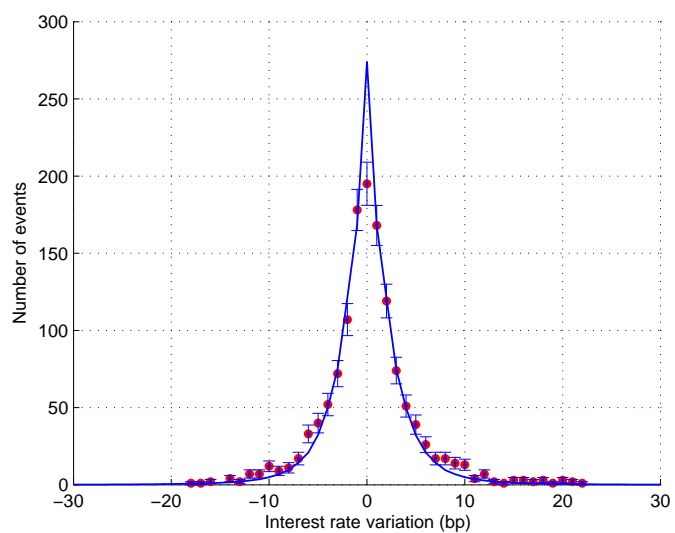
**Figure 6**

Maturity=[10], Initial Interest Rate  $\approx 7$ , Lag=1



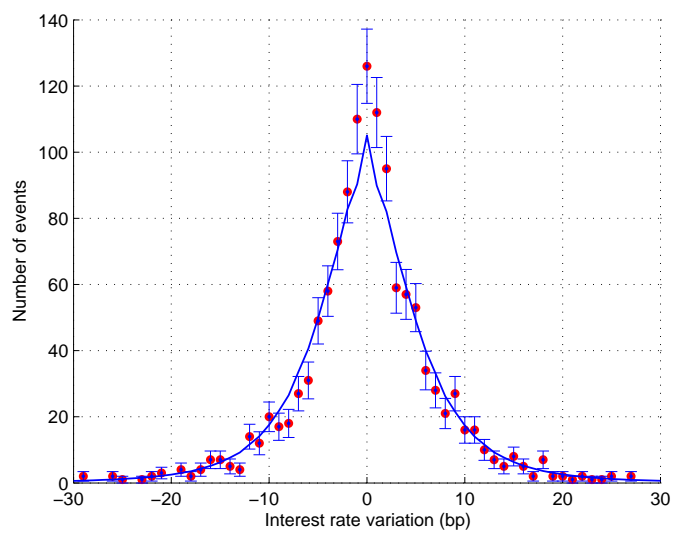
**Figure 7**

Maturity=[20], Initial Interest Rate  $\approx 5$ , Lag=1



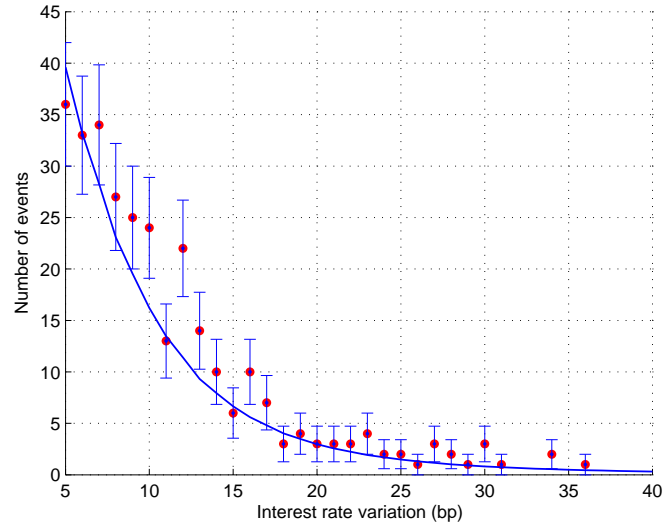
**Figure 8**

Maturity=[30], Initial Interest Rates  $\approx 9-10$ , Lag=1



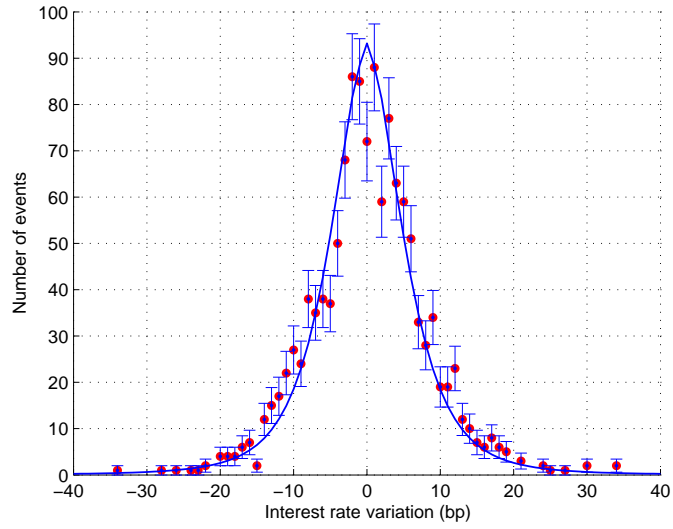
**Figure 9**

Maturity=[1], Initial Interest Rates  $\approx 9-10$ , Lag=1, Tail



**Figure 10**

Maturity=[5], Initial Interest Rate  $\approx 5$ , Lag=2



**Figure 11**

Maturity=[1], Initial Interest Rate  $\approx 7$ , Lag=2, Tail

